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Marginal and scalar solutions in cubic open string field theory

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Abstract: We find marginal and scalar solutions in cubic open string field theory by using left-right splitting properties of a delta function. The marginal solution represents a marginal deformation generated by a U(1) current, and it is a generalized solution of the Wilson lines one given by the present authors. The scalar solution has a well-defined universal Fock space expression, and it is expressed as a singular gauge transform of the trivial vacuum. The expanded theory around it is unable to be connected with the original theory by the string field redefinition.

Keywords: Bosonic Strings, D-branes.

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1. Introduction

Cubic open string field theory (CSFT) [1] has several classical solutions corresponding to the tachyon condensation [2, 3], tachyon lump [4, 5] and marginal deformations [6]. Since these solutions have been investigated numerically [7], it is necessary to obtain analytic solutions in order to study exact properties on classical solutions in CSFT. If an analytic solution of the tachyon vacuum exists, it is possible to prove Sen's conjectures on the tachyon condensation [8], or to construct new polynomial string field theory of closed strings, or to formulate vacuum string field theory completely [9, 10].

There are some attempts to obtain analytic solutions in CSFT [11, 12, 13]. In ref. [13], a Wilson lines solution and marginal tachyon lump solution are constructed without using a level truncation scheme with the Siegel gauge. As a result, it turns out that a branch cut singularity previously found in a marginal solution [6] should be gauge artifact because the analytic solution outside the Siegel gauge has no singularity. To construct the solutions, it is helpful to use the identity string field and left-right splitting properties of a delta function.

In this paper, we extend the Wilson lines solution to the ones representing general marginal deformations generated by a U(1) current. The splitting properties of a delta function, which is proved as a generalization of the ones in ref. [13], have an important role in the construction of marginal solutions. Due to the splitting properties of a delta function, some operators described by the left and right halves of a string become mutually commutable, and then the algebras of the operators are splitting into the left and right parts. Since CSFT is described by a half string formulation [1], the splitting algebras are powerful tools in order to find classical solutions in CSFT.

We also construct a scalar solution in terms of the splitting properties of a delta function, by which we find splitting algebras associated with the BRS current. The scalar

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solution has interesting features. It has a universal Fock space expression [3, 14], namely it is written by the matter Virasoro generators and ghost fields, since it is made of the BRS current, the ghost and the identity string field. In addition, the expanded theory around it can not be connected with the original theory by the string field redefinition. These properties are required for the tachyon vacuum solution.

Recently, it was proposed that classical solutions are constructed, which correspond to deformations of D-brane positions [15]. However, their solutions have singularity originated in a delta function, and so their Fock space expressions are ill-defined. On the other hand, our marginal and scalar solutions have well-defined Fock space expressions, and the marginal solutions contain the solution corresponding to deformations of D-brane positions.

This paper is organized as follows. In section 2, we prove the splitting properties of a delta function. In section 3, we construct general marginal solutions generated by a U(1) current. We construct a scalar solution which represents a non-trivial background in section 4. We give summary and discussions in section 5. We present some detail calculations in the appendix.

2. Splitting properties of delta function

We define a delta function as follows,

$$\delta(w,w') = \sum_{n=-\infty}^{\infty} w^{-n} w'^{n-1} = \sum_{n=-\infty}^{\infty} w'^{-n} w^{n-1}\,, \tag{2.1}$$

where w and w' are complex coordinates within a unit circle. If $f(w)$ has no pole except the origin, the delta function satisfies

$$f(w) = \oint_{C_0} \frac{dw'}{2\pi i}\delta(w,w')f(w')\,, \tag{2.2}$$

where C_0 denotes the contour which encircles the origin along the unit circle.

Let us consider integrations along left and right half unit circles as depicted in figure 1. We find that

$$\int_{C_{\rm L}} \frac{dw'}{2\pi i} w'^{n}\delta(w,w') = \frac{1}{2}w^n + \frac{1}{\pi}\sum_{k\neq n}\frac{1}{n-k}w^k \sin\left(\frac{(n-k)\pi}{2}\right). \tag{2.3}$$

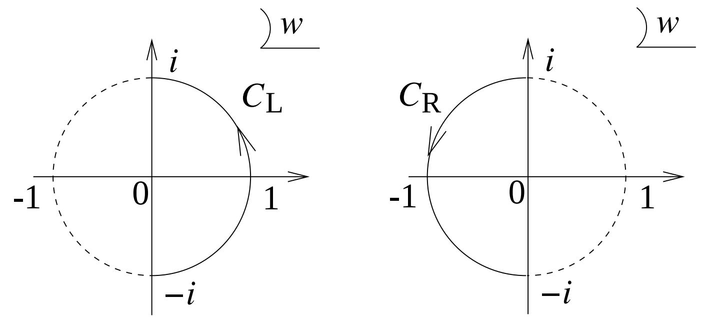

Figure 1: Contours $C_{\rm L}$ and $C_{\rm R}$ along the left and right halves of a string.

This implies that the delta function of eq. (2.1) is not a delta function if the integration path is the left half only. We perform the left integration of eq. (2.3) once more. We find that, if $m+n\neq 0$,

$$\begin{aligned}\int_{C_{\rm L}}\frac{dw}{2\pi i}w^{m-1}\int_{C_{\rm L}}\frac{dw'}{2\pi i}w'^{n}\delta(w,w') &= \frac{1}{\pi}\frac{1}{m+n}\sin\left(\frac{(m+n)\pi}{2}\right)+\\ &+\frac{1}{\pi^2}\sum_{\substack{k\neq 0,\\ m+n}}\frac{1}{k(k-m-n)}\sin\left(\frac{k\pi}{2}\right)\sin\left(\frac{(k-m-n)\pi}{2}\right),\end{aligned} \tag{2.4}$$

if $m+n=0$,

$$=\frac{1}{4}+\frac{1}{\pi^2}\sum_{k\neq 0}\frac{1}{k^2}\sin^2\left(\frac{k\pi}{2}\right). \tag{2.5}$$

Using the formula [13]

$$\sum_{k\neq 0,m}\frac{1}{k(k-m)}\sin\left(\frac{k\pi}{2}\right)\sin\left(\frac{(k-m)\pi}{2}\right)=\frac{\pi^2}{4}\delta_{m,0}\,, \tag{2.6}$$

we can calculate the infinite series and find the following equation,

$$\int_{C_{\rm L}}\frac{dw}{2\pi i}w^{m-1}\int_{C_{\rm L}}\frac{dw'}{2\pi i}w'^{n}\delta(w,w')=\int_{C_{\rm L}}\frac{dw}{2\pi i}w^{m+n-1}\,. \tag{2.7}$$

Therefore, if $f(w)$ and $g(w)$ have no pole except the origin, we find

$$\int_{C_{\rm L}}\frac{dw}{2\pi i}\int_{C_{\rm L}}\frac{dw'}{2\pi i}f(w)g(w')\delta(w,w')=\int_{C_{\rm L}}\frac{dw}{2\pi i}f(w)g(w)\,. \tag{2.8}$$

Thus, the delta function of eq. (2.1) behaves as a delta function in the double left integrations. From eqs. (2.2) and (2.8), other formulas are given by

$$\begin{aligned}\int_{C_{\rm R}}\frac{dw}{2\pi i}\int_{C_{\rm R}}\frac{dw'}{2\pi i}f(w)g(w')\delta(w,w') &= \int_{C_{\rm R}}\frac{dw}{2\pi i}f(w)g(w)\,,\\ \int_{C_{\rm L}}\frac{dw}{2\pi i}\int_{C_{\rm R}}\frac{dw'}{2\pi i}f(w)g(w')\delta(w,w') &= 0\,.\end{aligned} \tag{2.9}$$

These formulas of eqs. (2.8) and (2.9) are generalizations of the ones for the delta function with the Neumann boundary condition [13].

3. Marginal Solutions

We consider the ghost field $c(w)$, and a U(1) current $J(w)$, which is a dimension one primary field. These fields are expanded by

$$c(w)=\sum_n c_n w^{-n+1}\,,\qquad J(w)=\sum_n j_n w^{-n-1}\,. \tag{3.1}$$

The commutation relations of j_n are given by

$$[j_m,\,j_n]=m\delta_{m+n}\,. \tag{3.2}$$

From eq. (3.2), we find the commutation relation of $J(w)$ as

$$[J(w),\ J(w')] = -\partial_w \delta(w, w')\,. \tag{3.3}$$

Here, we introduce the following operators,

$$\begin{aligned}
V_{\mathrm{L}}(f) &= \int_{C_{\mathrm{L}}} \frac{dw}{2\pi i} f(w) c\, J(w)\,, \qquad & V_{\mathrm{R}}(f) &= \int_{C_{\mathrm{R}}} \frac{dw}{2\pi i} f(w) c\, J(w)\,, \\
C_{\mathrm{L}}(f) &= \int_{C_{\mathrm{L}}} \frac{dw}{2\pi i} f(w) c(w)\,, \qquad & C_{\mathrm{R}}(f) &= \int_{C_{\mathrm{R}}} \frac{dw}{2\pi i} f(w) c(w)\,,
\end{aligned} \tag{3.4}$$

where $f(w)$ is a holomorphic function within the unit circle except the origin, and satisfies $f(\pm i) = 0$. From the splitting formulas of the delta function, it follows that

$$\begin{aligned}
\{V_{\mathrm{L}}(f), V_{\mathrm{L}}(g)\} &= -\int_{C_{\mathrm{L}}} \frac{dw}{2\pi i} \int_{C_{\mathrm{L}}} \frac{dw'}{2\pi i} f(w) g(w') c(w) c(w') \partial_w \delta(w, w') \\
&= -\int_{C_{\mathrm{L}}} \frac{dw}{2\pi i} f(w) g(w) c\partial c(w) \\
&= -\{Q_{\mathrm{B}},\, C_{\mathrm{L}}(fg)\}\,,
\end{aligned} \tag{3.5}$$

where we have used $f(\pm i) = 0$ when we perform the partial integration. Similarly, we find that

$$\{V_{\mathrm{R}}(f), V_{\mathrm{R}}(g)\} = -\{Q_{\mathrm{B}},\, C_{\mathrm{R}}(fg)\}\,, \tag{3.6}$$

$$\{V_{\mathrm{L}}(f), V_{\mathrm{R}}(g)\} = 0\,. \tag{3.7}$$

Thus, we can obtain left-right splitting algebras in terms of the splitting property of the delta function.

Let us consider the functions $F_+^{(h)}(w)$ and $F_-^{(h)}(w)$ which, if we change the coordinate as $w' = -1/w$, transform into

$$\begin{aligned}
F_+^{(h)}(w') &= \left(\frac{dw}{dw'}\right)^h F_+^{(h)}(w)\,, \\
F_-^{(h)}(w') &= -\left(\frac{dw}{dw'}\right)^h F_-^{(h)}(w)\,.
\end{aligned} \tag{3.8}$$

$F_+^{(h)}(w)$ transforms as a dimension h field and $F_-^{(h)}(w)$ has a opposite sign compared with a dimension h field. Generally, the functions $F_\pm^{(h)}(w)$ are given by

$$\begin{aligned}
F_+^{(h)}(w) &= \sum_{n\geq 0} a_n v_n^{(h)}(w)\,, \\
F_-^{(h)}(w) &= \sum_{n\geq 0} b_n u_n^{(h)}(w)\,,
\end{aligned} \tag{3.9}$$

where $v_n^{(h)}$ and $u_n^{(h)}$ are defined by [16]

$$u_n^{(h)}(w) = w^{n-h} - (-1)^{n-h} w^{-n-h}\,, \qquad v_n^{(h)}(w) = w^{n-h} + (-1)^{n-h} w^{-n-h}\,. \tag{3.10}$$

The N-strings vertex of a midpoint interaction is defined by gluing the boundaries $|w_i| = 1$ $(i = 1, 2, \ldots, N)$ of N unit disks with the identifications [17, 18],

$$w_i\, w_{i+1} = -1\,, \qquad \text{for } |w_i| = 1,\ \mathrm{Re}\, w_i \leq 1\,, \tag{3.11}$$

where w_{N+1} denotes w_1. If $\phi(w)$ is a dimension h primary field, one forms of $dwF_+^{(-h+1)}\phi$ and $dwF_-^{(-h+1)}\phi$ are transformed as

$$\begin{aligned} dw_{i+1}F_+^{(-h+1)}(w_{i+1})\phi(w_{i+1}) &= dw_iF_+^{(-h+1)}(w_i)\phi(w_i)\,, \\ dw_{i+1}F_-^{(-h+1)}(w_{i+1})\phi(w_{i+1}) &= -dw_iF_-^{(-h+1)}(w_i)\phi(w_i)\,. \end{aligned} \tag{3.12}$$

Then, considering $cJ(w)$ and $c(w)$ as the primary fields, we obtain the following equations related to star product:

$$(V_{\mathrm{R}}(F_-^{(1)})A) * B = (-)^{|A|}A * (V_{\mathrm{L}}(F_-^{(1)})B)\,, \tag{3.13}$$

$$(C_{\mathrm{R}}(F_+^{(2)})A) * B = -(-)^{|A|}A * (C_{\mathrm{L}}(F_+^{(2)})B)\,, \tag{3.14}$$

where A and B are arbitrary string fields, and $|A|$ is 0 if A is Grassmann even and 1 if it is odd. Similarly, we find that

$$V_{\mathrm{L}}(F_-^{(1)})I = V_{\mathrm{R}}(F_-^{(1)})I\,, \qquad C_{\mathrm{L}}(F_+^{(2)})I = -C_{\mathrm{R}}(F_+^{(2)})I\,, \tag{3.15}$$

where I denotes the identity string field. The function $F_+^{(2)}(w)$ must satisfy $F_+^{(2)}(\pm i) = 0$, because the ghost $c(w)$ has the midpoint singularity on the identity string field, which is evaluated by the oscillator expression as [13]

$$c(w)\,|I\rangle = \left[-c_0\frac{w-w^3}{1+w^2} + c_1\frac{w^2}{1+w^2} + c_{-1}\frac{1+w^2+w^4}{1+w^2} + \sum_{n\geq 2} c_{-n}v_n^{(-1)}(w)\right]|I\rangle\,. \tag{3.16}$$

Now, we obtain a classical solution associated with the U(1) current,

$$\Psi_{\mathrm{m}} = \mathrm{diag}\left(a_iV_{\mathrm{L}}(F_-^{(1)}) + \frac{1}{2}a_i^2\,C_{\mathrm{L}}(F_-^{(1)^2})\right)I, \tag{3.17}$$

where a_i are parameters and i corresponds to the Chan-Paton index. From eqs. (3.5), (3.13), (3.14) and (3.15), it follows that

$$\begin{aligned} Q_{\mathrm{B}}\Psi_{\mathrm{m}} &= \frac{1}{2}a_i^2\left\{Q_{\mathrm{B}},\, C_{\mathrm{L}}(F_-^{(1)^2})\right\}I\,, \\ \Psi_{\mathrm{m}} * \Psi_{\mathrm{m}} &= a_i^2V_{\mathrm{L}}(F_-^{(1)})I * V_{\mathrm{L}}(F_-^{(1)})I + \frac{1}{2}a_i^2V_{\mathrm{L}}(F_-^{(1)})I * C_{\mathrm{L}}(F_-^{(1)^2})I + \\ &\quad + \frac{1}{2}a_i^2C_{\mathrm{L}}(F_-^{(1)^2})I * V_{\mathrm{L}}(F_-^{(1)})I + \frac{1}{4}a_i^4C_{\mathrm{L}}(F_-^{(1)^2})I * C_{\mathrm{L}}(F_-^{(1)^2})I \\ &= \frac{1}{2}a_i^2\left\{V_{\mathrm{L}}(F_-^{(1)}),\, V_{\mathrm{L}}(F_-^{(1)})\right\}I \\ &= -\frac{1}{2}a_i^2\left\{Q_{\mathrm{B}},\, C_{\mathrm{L}}(F_-^{(1)^2})\right\}I\,. \end{aligned} \tag{3.18}$$

Then, the classical solution satisfies the equation of motion, $Q_{\rm B}\Psi_{\rm m}+\Psi_{\rm m}*\Psi_{\rm m}=0$. The solution for $F_-^{(1)}(w)=u_0^{(1)}(w)+u_2^{(1)}(w)$ are give by ref. [13]. Since the classical solution of eq. (3.17) dose not refer to any boundary condition, it is a generalization of the solution given by ref. [13].

If we expand the string field around the classical solution, the BRS charge becomes

$$Q'_{\rm B}=Q_{\rm B}+a_iV_{\rm L}(F_-^{(1)})-a_jV_{\rm R}(F_-^{(1)})+\frac{1}{2}a_i^2C_{\rm L}(F_-^{(1)^2})+\frac{1}{2}a_j^2C_{\rm R}(F_-^{(1)^2})\,, \tag{3.19}$$

which operates to each $(i,\,j)$-component of the string field Ψ_{ij}. Supposed that there is the dimension 'zero' field $\varphi(w)$ which satisfies

$$\big[\varphi(w),\,J(w')\big]=i\,\delta(w,w')\,,\qquad [Q_{\rm B},\,\varphi(w)]=-ic\,J(w)\,. \tag{3.20}$$

For the operator φ, we introduce the left and right integrated operators,

$$\Phi_{\rm L}(f)=\int_{C_{\rm L}}\frac{dw}{2\pi i}f(w)\varphi(w)\,,\qquad \Phi_{\rm R}(f)=\int_{C_{\rm R}}\frac{dw}{2\pi i}f(w)\varphi(w)\,. \tag{3.21}$$

From eq. (3.20), it follows that

$$[\Phi_{\rm L}(f),\,Q_{\rm B}]=iV_{\rm L}(f)\,,\qquad [\Phi_{\rm L}(f),\,V_{\rm L}(g)]=iC_{\rm L}(fg)\,. \tag{3.22}$$

By these commutation relations, the shifted BRS charge can be written as

$$Q'_{\rm B}=e^{-B_{ij}(F_-^{(1)})}Q_{\rm B}e^{B_{ij}(F_-^{(1)})}\,, \tag{3.23}$$

where B is defined by

$$B_{ij}(F_-^{(1)})=ia_i\Phi_{\rm L}(F_-^{(1)})-ia_j\Phi_{\rm R}(F_-^{(1)})\,. \tag{3.24}$$

Therefore, it seems that the shifted theory is transformed into the original theory by the redefinition of the string field and the classical solution is trivial. However, it is true only if the field $\varphi(w)$ exists and the redefinition is well-defined for the zero-mode part of φ. For example, we consider $J\sim i\partial X$ and $\varphi\sim X$. In this case, if the direction X is compactified, the redefinition is ill-defined for the zero-mode of X. Indeed, the shifted theory describes strings in the Wilson lines background, and the classical solution corresponds to the condensation of the gauge fields [13].

Using the operator $\Phi_{\rm L}$, the classical solution can be rewritten as

$$\Psi_{\rm m}=\exp\Big(-i\,a_i\Phi_{\rm L}(F_-^{(1)})I\Big)*Q_{\rm B}\,\exp\Big(i\,a_i\Phi_{\rm L}(F_-^{(1)})I\Big)\,. \tag{3.25}$$

It implies that the classical solution is a gauge transform of zero string field, namely it is pure gauge. However, as well as the string field redefinition, there are cases in which the gauge transformation is ill-defined because of the zero-mode of Φ_L. For the Wilson lines solution, the classical solution is locally pure gauge, but it is globally non-trivial configuration, analogously to field theoretical situations.

Let us consider the potential height $S[\Psi_{\rm m}]$ at the classical solution. Since the classical solution has parameters a_i, it follows that

$$\frac{d}{da_i}S[\Psi_{\rm m}] = \frac{2}{g}\int (Q_{\rm B}\Psi_{\rm m} + \Psi_{\rm m} * \Psi_{\rm m}) * \frac{d\Psi_{\rm m}}{da_i} = 0\,. \tag{3.26}$$

Then, we find that $S[\Psi_{\rm m}(a_i)] = S[\Psi_{\rm m}(a_i = 0)] = 0$ [19]. Thus, the potential height is always zero and a_i correspond to marginal parameters.

We can construct classical solutions by using $F_+^{(1)}$ instead of $F_-^{(1)}$. However, $\Phi_{\rm L}(F_+^{(1)})$ can not involve the zero mode of φ because of $v_0^{(1)}(w) = 0$. Therefore, the solutions correspond to pure gauge and physically trivial solutions.

4. Scalar solutions

The BRS current is defined by

$$J_{\rm B}(w) = c\left(T_X + \frac{1}{2}T_{\rm gh}\right)(w) + \frac{3}{2}\partial^2 c(w)\,, \tag{4.1}$$

where $T_X(w)$ and $T_{\rm gh}(w)$ denote the energy momentum tensors of string coordinates and reparametrization ghosts, respectively [20, 21]. The operator product expansions (OPEs) of the BRS current and the ghost field are given by

$$\begin{aligned}
J_{\rm B}(w)J_{\rm B}(w') &= \frac{-(d-18)/2}{(w-w')^3}c\partial c(w') + \frac{-(d-18)/4}{(w-w')^2}c\partial^2 c(w') - \frac{(d-26)/12}{w-w'}c\partial^3 c(w') + \cdots \\
&= \frac{-4}{(w-w')^3}c\partial c(w') + \frac{-2}{(w-w')^2}c\partial^2 c(w') + \cdots\,, \\
J_{\rm B}(w)c(w') &= \frac{1}{w-w'}c\partial c(w') + \cdots\,,
\end{aligned} \tag{4.2}$$

where $d = 26$ is the matter central charge of the conformal field theory. We can expand the BRS current and the ghost field using oscillation modes,

$$\begin{aligned}
J_{\rm B}(w) &= \sum_{n=-\infty}^{\infty} Q_n w^{-n-1}\,, \\
c(w) &= \sum_{n=-\infty}^{\infty} c_n w^{-n+1}\,.
\end{aligned} \tag{4.3}$$

Since $\{Q_{\rm B},\, c(w)\} = c\partial c(w)$, the OPEs of eq. (4.2) can be rewritten in the form of anti-commutation relations of these oscillators,

$$\{Q_m,\, Q_n\} = 2mn\{Q_{\rm B},\, c_{m+n}\}\,, \qquad \{Q_m,\, c_n\} = \{Q_{\rm B},\, c_{m+n}\}\,. \tag{4.4}$$

From eq. (4.4), we find the anti-commutation relation of the BRS current and the ghost,

$$\begin{aligned}
\{J_{\rm B}(w),\, J_{\rm B}(w')\} &= \left\{Q_{\rm B},\, 2\,\partial_w\partial_{w'}\left(c(w)\delta(w,\, w')\right)\right\}, \\
\{J_{\rm B}(w),\, c(w')\} &= \{Q_{\rm B},\, c(w)\delta(w,\, w')\}\,.
\end{aligned} \tag{4.5}$$

We now define the following operators,

$$Q_{\rm L}(f)=\int_{C_{\rm L}}\frac{dw}{2\pi i}f(w)J_{\rm B}(w)\,,\qquad Q_{\rm R}(f)=\int_{C_{\rm R}}\frac{dw}{2\pi i}f(w)J_{\rm B}(w)\,,\tag{4.6}$$

where $f(w)$ is a holomorphic function within the unit circle except the origin, and, in addition, its values at the midpoint is zero, $f(\pm i)=0$. From eq. (4.5), we can calculate the anti-commutation relation of the operators as follows,

$$\begin{aligned}\{Q_{\rm L}(f),\,Q_{\rm L}(g)\}&=\left\{Q_{\rm B},\,\int_{C_{\rm L}}\frac{dw}{2\pi i}\int_{C_{\rm L}}\frac{dw'}{2\pi i}f(w)g(w')2\,\partial_w\partial_{w'}\left(c(w)\delta(w,\,w')\right)\right\}\\&=2\left\{Q_{\rm B},\,\int_{C_{\rm L}}\frac{dw}{2\pi i}\int_{C_{\rm L}}\frac{dw'}{2\pi i}\partial f(w)\partial g(w')c(w)\delta(w,\,w')\right\},\end{aligned}\tag{4.7}$$

where surface terms are vanished due to $f(\pm i)=g(\pm i)=0$. Using the delta function formula, we find that

$$\{Q_{\rm L}(f),\,Q_{\rm L}(g)\}=2\{Q_{\rm B},\,C_{\rm L}(\partial f\partial g)\}\,.\tag{4.8}$$

Similarly, other anti-commutation relations are given by

$$\begin{aligned}\{Q_{\rm R}(f),\,Q_{\rm R}(g)\}&=2\,\{Q_{\rm B},\,C_{\rm R}(\partial f\partial g)\}\,,\\\{Q_{\rm L}(f),\,C_{\rm L}(g)\}&=\{Q_{\rm B},\,C_{\rm L}(fg)\}\,,\\\{Q_{\rm R}(f),\,C_{\rm R}(g)\}&=\{Q_{\rm B},\,C_{\rm R}(fg)\}\,,\\\{Q_{\rm L}(f),\,Q_{\rm R}(g)\}&=\{Q_{\rm L}(f),\,C_{\rm R}(g)\}=\{Q_{\rm R}(f),\,C_{\rm L}(g)\}=0.\end{aligned}\tag{4.9}$$

Thus, we can obtain the splitting algebras associated with the BRS current by using the splitting properties of the delta function.

We consider the properties of the $Q_{\rm L(R)}$ related to star product and the identity string field. Since $dw_iF_+^{(0)}(w_i)J_{\rm B}(w_i)$ is a globally defined one form in the gluing N-strings surface, we obtain a similar equation to eqs. (3.13) and (3.14),

$$\left(Q_{\rm R}(F_+^{(0)})A\right)*B=-(-)^{|A|}A*\left(Q_{\rm L}(F_+^{(0)})B\right).\tag{4.10}$$

Similarly, we find that

$$Q_{\rm L}(F_+^{(0)})I=-Q_{\rm R}(F_+^{(0)})I\,.\tag{4.11}$$

Now, we can show that a classical solution is given by

$$\Psi_0=Q_{\rm L}(F_+^{(0)})I+C_{\rm L}(G_+^{(2)})I\,,\tag{4.12}$$

where $G_+^{(2)}(w)$ is

$$G_+^{(2)}(w)=-\frac{\left(\partial F_+^{(0)}(w)\right)^2}{1+F_+^{(0)}(w)}\,.\tag{4.13}$$

$G_+^{(2)}(\pm i)$ must be zero in order to cancel the midpoint singularity of the ghost on the identity, as in the case of the marginal solutions. Indeed, from eqs. (3.14), (4.8), (4.9), (4.10) and (4.11), we find that the classical equation satisfies the equation of motion:

$$\begin{aligned}Q_{\rm B}\Psi_0&=\{Q_{\rm B},\,C_{\rm L}(G_+^{(2)})\}I\,,\\\Psi_0*\Psi_0&=\{Q_{\rm B},\,C_{\rm L}((\partial F_+^{(0)})^2+F_+^{(0)}G_+^{(2)})\}I\,.\end{aligned}\tag{4.14}$$

Then, it follows that $Q_{\rm B}\Psi_0+\Psi_0*\Psi_0=0$.

If we expand the string field around the classical solution, the shifted theory has the following BRS charge,

$$Q'_{\mathrm{B}} = Q_{\mathrm{B}} + Q(F_+^{(0)}) + C(G_+^{(2)}) , \tag{4.15}$$

where we define

$$Q(f) = Q_{\mathrm{L}}(f) + Q_{\mathrm{R}}(f) , \qquad C(f) = C_{\mathrm{L}}(f) + C_{\mathrm{R}}(f) . \tag{4.16}$$

If we take $F_+^{(0)}(w) = \exp(h(w)) - 1$, the classical solution and the shifted BRS charge are rewritten as

$$\Psi_0 = Q_{\mathrm{L}}(e^h - 1)I - C_{\mathrm{L}}\left((\partial h)^2 e^h\right) I , \tag{4.17}$$

$$Q'_{\mathrm{B}} = Q(e^h) - C\left((\partial h)^2 e^h\right) . \tag{4.18}$$

Let us consider the redefinition of the string field. The ghost number currents are given by

$$J_{\mathrm{gh}}(w) = cb(w) , \tag{4.19}$$

where $c(w)$ and $b(w)$ are ghost and anti-ghost fields, respectively. The OPEs of the ghost current with the BRS current and with the ghost field are given by

$$J_{\mathrm{gh}}(w)J_{\mathrm{B}}(w') = \frac{4}{(w-w')^3}c(w') + \frac{2}{(w-w')^2}\partial c(w') + \frac{1}{(w-w')^2}J_{\mathrm{B}}(w') + \cdots , \tag{4.20}$$

$$J_{\mathrm{gh}}(w)c(w') = \frac{1}{w-w'}c(w') + \cdots . \tag{4.21}$$

We introduce an operator for the function $f(w)$ which is holomorphic except the origin,

$$q(f) = \oint \frac{dw}{2\pi i} f(w) J_{\mathrm{gh}}(w) . \tag{4.22}$$

The OPEs of eqs. (4.20) and (4.21) give the following commutation relations,

$$[q(f),\, Q(g)] = Q(fg) - 2\,C(\partial f \partial g) , \tag{4.23}$$

$$[q(f),\, C(g)] = C(fg) . \tag{4.24}$$

From the commutation relations of eqs. (4.23) and (4.24), we find that, through the transformation generated by $q(f)$, the BRS charge becomes

$$\begin{aligned} e^{q(f)}\, Q_{\mathrm{B}}\, e^{-q(f)} &= Q_{\mathrm{B}} + [q(f),\, Q_{\mathrm{B}}] + \frac{1}{2!}[q(f),\, [q(f),\, Q_{\mathrm{B}}]] + \cdots \\ &= Q_{\mathrm{B}} + Q(f) + \frac{1}{2!}\left\{Q(f^2) - 2C((\partial f)^2)\right\} + \cdots \\ &= Q(e^f) - C\left((\partial f)^2 e^f\right) . \end{aligned} \tag{4.25}$$

Therefore, if the string field Ψ is redefined as $\Psi = e^{q(h)}\Psi'$, the shifted BRS charge is transformed into the original BRS charge.

In order to interpret the background of the redefined theory, let us consider the conservation law of $q(h)$ on the N-strings vertex. The gluing N-strings surface can be transformed into the whole complex z-plane by the mapping

$$z = e^{\frac{2\pi(1-k)i}{N}} \left(\frac{1+iw_k}{1-iw_k}\right)^{2/N}, \qquad (k=1,\ldots,N)\,. \tag{4.26}$$

Here, $\exp(2\pi(1-k)i/N)$ $(k=1,\ldots,N)$ correspond to the N punctures in the z-plane, which represent N strings insertions, and the origin and the infinity in the z plane correspond to the midpoints of the N strings. Since $h(w)$ is an analytic scalar and $F_+^{(0)}(\pm i)=0$, we find that $h(z=0)=h(z=\infty)=0$. Therefore, the conservation law in the z plane is given by [16]

$$\langle V_N| \sum_{k=1}^{N} \int_{C_k} \frac{dz}{2\pi i} h(z) J_{\rm gh}(z) = 0\,, \tag{4.27}$$

where the contour C_k encircles the puncture at the k-string in the z plane. The anomalous term at the infinity vanishes due to $h(\infty)=0$. We can express the contour integral around the k-string's puncture in terms of the local coordinates w_k. Since the transformation low of the ghost number current $J_{\rm gh}$ is given by

$$\frac{dz}{dw} J_{\rm gh}(z) = J_{\rm gh}(w) + \frac{3}{2}\frac{d^2z}{dw^2}\left(\frac{dz}{dw}\right)^{-1}, \tag{4.28}$$

we obtain the following identity [16],

$$\begin{aligned}
\langle V_N| \sum_{k=1}^{N} q^{(k)}(h) &= \langle V_N| \sum_{k=1}^{N} \oint_{C_0} \frac{dw_k}{2\pi i}\, h(w_k)\, J_{\rm gh}(w_k) = \kappa_N(h)\, \langle V_N|\,, \\
\kappa_N(h) &= -\frac{3}{2}\sum_{k=1}^{N} \oint_{C_0} \frac{dw_k}{2\pi i}\, h(w_k)\, \frac{d^2 z}{d{w_k}^2}\left(\frac{dz}{dw_k}\right)^{-1},
\end{aligned} \tag{4.29}$$

where C_0 denotes the contour around the origin. From eq. (4.26), we find that

$$\begin{aligned}
\kappa_N(h) &= -\frac{3}{2}\sum_{k=1}^{N} \oint_{C_0} \frac{dw_k}{2\pi i}\, h(w_k)\left(\frac{4i}{N}\frac{1}{1+w_k^2} - \frac{2w_k}{1+w_k^2}\right) \\
&= -6i \oint_{C_0} \frac{dw}{2\pi i}\, h(w)\, \frac{1}{1+w^2} + 3N \oint_{C_0} \frac{dw}{2\pi i}\, h(w)\, \frac{w}{1+w^2}\,.
\end{aligned} \tag{4.30}$$

The shifted action around the classical solution is given by

$$S = \frac{1}{g}\int\left(\Psi * Q'_{\rm B}\Psi + \frac{2}{3}\Psi * \Psi * \Psi\right). \tag{4.31}$$

The action is made of a reflector and a three strings vertex, which are 2-strings and 3-strings vertices, respectively. If we redefine the string field as $\Psi = e^{q(h)}\Psi'$, the action becomes[1]

$$S = \frac{1}{g}\int\left(e^{\kappa_2(h)}\,\Psi' * Q_{\rm B}\Psi' + \frac{2}{3}\,e^{\kappa_3(h)}\,\Psi' * \Psi' * \Psi'\right). \tag{4.32}$$

[1] In ref. [22], the classical solution was constructed for a particular function $h(w)$. However, κ_2 was missed when it was physically interpreted. So, the conclusion has changed in this paper.

Supposed that the first term of eq. (4.30) has non-zero value. Since $\kappa_2/2 \neq \kappa_3/3$, we can not absorb these factors by rescaling the string field, and the shifted theory becomes the theory with the original BRS charge and the different coupling constant. So, the classical solution may represent the dilaton condensation. If the first term vanishes, the shifted theory become the original one by rescaling the string field, and so the classical solution may correspond to a pure gauge solution. However, each cases should be investigated more carefully, because there is a possibility that the redefinition itself is ill-defined.

Expanding the ghost current as $J_{\rm gh}(w) = \sum_n q_n w^{-n-1}$, we can write the operator $q(h)$ as

$$q(h) = q_0 \oint \frac{dw}{2\pi i} w^{-1} h(w) + q^{(+)}(h) + q^{(-)}(h)\,, \tag{4.33}$$

where $q^{(+)}(h)$ and $q^{(-)}(h)$ which correspond to positive and negative mode parts of $q(h)$ are given by

$$\begin{aligned} q^{(+)}(h) &= \sum_{n\geq 1}^{\infty} q_n \oint \frac{dw}{2\pi i} w^{-n-1} h(w)\,, \\ q^{(-)}(h) &= \sum_{n\geq 1}^{\infty} q_{-n} \oint \frac{dw}{2\pi i} w^{n-1} h(w)\,. \end{aligned} \tag{4.34}$$

The operator $e^{q(h)}$ is rewritten by the 'normal ordered' form as follows,

$$e^{q(h)} = \exp\left(q_0 \oint \frac{dw}{2\pi i} w^{-1} h(w)\right) \exp\left(\frac{1}{2}\left[q^{(+)}(h),\, q^{(-)}(h)\right]\right) e^{q^{(-)}(h)} e^{q^{(+)}(h)}\,. \tag{4.35}$$

If the commutator of $q^{(\pm)}(h)$ has singularity, the operator $e^{q(h)}$ is ill-defined and we can not redefine the string field by $\Psi = e^{q(h)}\Psi'$. Then, using such a singular function $h(w)$, we can obtain a non-trivial classical solution. From the OPE of the ghost number currents

$$J_{\rm gh}(w) J_{\rm gh}(w') = \frac{1}{(w-w')^2} + \cdots\,, \tag{4.36}$$

the oscillators q_n satisfy $[q_m,\, q_n] = m\delta_{m+n}$. Then, we can write generally the commutator of $q^{(\pm)}(h)$ as

$$\left[q^{(+)}(h),\, q^{(-)}(h)\right] = \oint \frac{dw}{2\pi i} \oint \frac{dw'}{2\pi i} \frac{1}{(w-w')^2} h(w) h(w')\,. \tag{4.37}$$

For example, let us consider the classical solution for

$$h_a(w) = \log\left(1 + \frac{a}{2}\left(w + \frac{1}{w}\right)^2\right), \tag{4.38}$$

where a is a real parameter which is larger than or equal to $-1/2$. For this $h_a(w)$, $F_+^{(0)}(w)$ and $G_+^{(2)}(w)$ are given by

$$\begin{aligned} F_+^{(0)}(w) &= e^{h(w)} - 1 = \frac{a}{2}\left(w + \frac{1}{w}\right)^2 = \frac{a}{2}\left(v_0^{(0)}(w) + v_2^{(0)}(w)\right), \\ G_+^{(2)}(w) &= -(\partial h(w))^2 e^{h(w)} = -a^2 w^{-2} \frac{\left(w^2 - 1/w^2\right)^2}{1 + \frac{a}{2}\left(w + 1/w\right)^2}\,. \end{aligned} \tag{4.39}$$

Indeed, we find that $F_+^{(0)}(\pm i) = 0$ and $G_+^{(2)}(\pm i) = 0$, and this $h_a(w)$ gives the classical solution by eq. (4.17). The function $h_a(w)$ has the Laurent expansion as follows (see in appendix A),

$$\begin{aligned} h_a(w) &= -\log(1-Z(a))^2 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} Z(a)^n \left(w^{2n} + \frac{1}{w^{2n}} \right) , \\ Z(a) &= \frac{1+a-\sqrt{1+2a}}{a} . \end{aligned} \tag{4.40}$$

Using this expansion and eq. (4.30), we can evaluate $\kappa_N(h_a)$ as

$$\kappa_N(h_a) = -3N \log(1-Z(a)) . \tag{4.41}$$

Here, the first term of eq. (4.30) vanishes. Therefore, we may naively expect that the shifted action reduces the original one and the classical solution might be pure gauge.

However, we must consider the string field redefinition in detail as general discussion. From eq. (4.40), the operator $q(h_a)$ can be expressed by

$$\begin{aligned} q(h_a) &= -q_0 \log(1-Z(a))^2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (q_{2n} + q_{-2n}) Z(a)^n \\ &= -q_0 \log(1-Z(a))^2 + q^{(+)}(h_a) + q^{(-)}(h_a) , \end{aligned} \tag{4.42}$$

where $q^{(+)}$ and $q^{(-)}$ denote the positive and negative modes part of $q(h_a)$;

$$q^{(+)}(h_a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} q_{2n} Z(a)^n , \qquad q^{(-)}(h_a) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} q_{-2n} Z(a)^n . \tag{4.43}$$

Using $[q_m, q_n] = m\delta_{m+n}$, we can evaluate the commutation relation of $q^{(\pm)}$ as follows,

$$[q^{(+)}(h_a), q^{(-)}(h_a)] = 2 \sum_{n=1}^{\infty} \frac{1}{n} Z(a)^{2n} = -2 \log(1-Z(a)^2) . \tag{4.44}$$

Then, we can rewrite the operator $e^{q(h_a)}$ by the 'normal ordered' form

$$e^{q(h_a)} = \left(1-Z(a)^2\right)^{-1} \exp\left(-q_0 \log(1-Z(a))^2\right) e^{q^{(-)}(h_a)} e^{q^{(+)}(h_a)} . \tag{4.45}$$

For $a > -1/2$, this operator is well-defined since $|Z(a)| < 1$. Therefore, the classical solution for $a > -1/2$ should be pure gauge solution. However, in the case of $Z(a = -1/2) = -1$, this operator $e^{q(h_a)}$ has singularity and the string field redefinition is ill-defined. Thus, we can obtain a non-trivial classical solution for $a = -1/2$.

In the case of $a = -1/2$, the classical solution is given by

$$\Psi_0 = Q_{\rm L} \left(-\frac{1}{4} \left(w + \frac{1}{w} \right)^2 \right) I + C_{\rm L} \left(w^{-2} \left(w + \frac{1}{w} \right)^2 \right) I . \tag{4.46}$$

Each term of eq. (4.46) has a well-defined Fock space expression as follows,

$$
\begin{aligned}
Q_{\rm L}\left(-\frac{1}{4}\left(w+\frac{1}{w}\right)^2\right)|I\rangle &= -\sum_{n=0}^{\infty}\frac{(-1)^n}{2\pi}\left(\frac{2}{2n+1}-\frac{1}{2n+3}-\frac{1}{2n-1}\right)Q_{-2n-1}\,|I\rangle\ , \\
C_{\rm L}\left(w^{-2}\left(w+\frac{1}{w}\right)^2\right)|I\rangle &= \left[\frac{2}{\pi}c_1+\frac{10}{3\pi}c_{-1}+\right. \\
&\qquad \left.+\,2\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi}\left(\frac{2}{2n+1}-\frac{1}{2n+3}-\frac{1}{2n-1}\right)c_{-2n-1}\right]|I\rangle\ ,
\end{aligned}
\tag{4.47}
$$

where use has been made of eq. (3.16) and

$$
J_{\rm B}(w)\,|I\rangle = \sum_{n=1}^{\infty} Q_{-n}\,v_n^{(1)}(w)\,|I\rangle\ . \tag{4.48}
$$

If we expand the string field around the classical solution, the shifted BRS charge is given by

$$
Q'_{\rm B} = \frac{1}{2}Q_{\rm B} - \frac{1}{4}\left(Q_2+Q_{-2}\right) + 2c_0 + c_2 + c_{-2}\,. \tag{4.49}
$$

The BRS current is written by the matter Virasoro generators and the ghost fields, and the identity string field is given by the Virasoro generators [16]. Therefore, the scalar solution has a universal expression for arbitrary backgrounds.

Finally, we show that the classical solution for $a=-1/2$ is generated by a singular gauge transformation. From the OPE of eqs. (4.20) and (4.21), we can calculate the commutation relations of $J_{\rm gh}$ with $J_{\rm B}$ and c,

$$
\begin{aligned}
[J_{\rm gh}(w),\, J_{\rm B}(w')] &= J_{\rm B}(w)\delta(w,w') - 2\partial_w\partial_{w'}\left(c(w)\delta(w,w')\right), \\
[J_{\rm gh}(w),\, c(w')] &= c(w)\delta(w,w')\,.
\end{aligned}
\tag{4.50}
$$

We introduce the operators corresponding to the left and right pieces of $q(h)$,

$$
q_{\rm L}(f) = \int_{C_{\rm L}}\frac{dw}{2\pi i}f(w)J_{\rm gh}(w)\,, \qquad q_{\rm R}(f) = \int_{C_{\rm R}}\frac{dw}{2\pi i}f(w)J_{\rm gh}(w)\,, \tag{4.51}
$$

where we impose $f(\pm i)=0$. From eq. (4.50), these operators satisfy

$$
\begin{aligned}
[q_{\rm L}(f),\, Q_{\rm L}(g)] &= Q_{\rm L}(fg) - 2C_{\rm L}(\partial f\partial g)\,, \\
[q_{\rm L}(f),\, C_{\rm L}(g)] &= C_{\rm L}(fg)\,,
\end{aligned}
\tag{4.52}
$$

and the left and right operators are commutable each other. From eq. (4.29), we find as the case of $N=1$

$$
\langle I|\left(q_{\rm L}(h_a)+q_{\rm R}(h_a)\right) = \kappa_1(h_a)\,\langle I|\,, \qquad \kappa_1(h_a) = -3\log(1-Z(a))\,. \tag{4.53}
$$

From the anomalous transformation law of eq. (4.28), we find that

$$
\langle V_N|\left(q_{\rm L}^{(r+1)}(h_a)+q_{\rm R}^{(r)}(h_a)\right) = -3\int_{C_{\rm L}}\frac{dw}{2\pi i}w^{-1}h_a(w)\,\langle V_N|\ , \tag{4.54}
$$

where r represents the Hilbert space of the r-th string. Using eq. (4.40), we can evaluate the coefficient in the right hand side as

$$-3\int_{C_{\rm L}}\frac{dw}{2\pi i}w^{-1}h_a(w)=-3\log(1-Z(a))=\kappa_1(h_a)\,. \tag{4.55}$$

From eqs. (4.52), (4.53) and (4.54), the classical solution of eq. (4.17) for $h_a(w)$ is expressed as the gauge transform of zero string field,

$$\Psi_0=\exp\left(q_{\rm L}(h_a)I\right)*Q_{\rm B}\exp\left(-q_{\rm L}(h_a)I\right). \tag{4.56}$$

Let us consider the case of $a=-1/2$. Using eq. (4.40), we can write the operator $q_{\rm L}(h_{-1/2})$ by the mode expression,

$$q_{\rm L}(h_{-1/2})=-q_0\log 2+q_{\rm L}^{(+)}(h_{-1/2})+q_{\rm L}^{(-)}(h_{-1/2})\,, \tag{4.57}$$

where we define

$$q_{\rm L}^{(+)}(h_{-1/2}) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}q_{2n}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\beta\Big(n-\frac{1}{2}\Big)+\beta\Big(-n+\frac{1}{2}\Big)\right)q_{2n-1} \tag{4.58}$$

$$q_{\rm L}^{(-)}(h_{-1/2}) = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}q_{-2n}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\beta\Big(n-\frac{1}{2}\Big)+\beta\Big(-n+\frac{1}{2}\Big)\right)q_{-2n+1}. \tag{4.59}$$

Here, $\beta(z)$ is defined by

$$\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{z+n}\,, \tag{4.60}$$

and it is written by digamma function $\psi(z)=d\log\Gamma(z)/dz$,

$$\beta(z)=\frac{1}{2}\left(\psi\left(\frac{z+1}{2}\right)-\psi\left(\frac{z}{2}\right)\right). \tag{4.61}$$

Using eqs. (4.58) and (4.59), we can calculate the commutator of $q_{\rm L}^{(\pm)}$ as follows,

$$\left[q_{\rm L}^{(+)}(h_{-1/2}),\,q_{\rm L}^{(-)}(h_{-1/2})\right]=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\beta\Big(n-\frac{1}{2}\Big)+\beta\Big(-n+\frac{1}{2}\Big)\right)^2. \tag{4.62}$$

Thus, the commutator of $q_{\rm L}^{(\pm)}$ has divergence as well as the one of $q^{(\pm)}$. Rewriting $\exp(\pm q_{\rm L}I)$ as

$$\begin{aligned}\exp\left(\pm q_{\rm L}(h_{-1/2})I\right) &= \exp\left(\pm q_{\rm L}(h_{-1/2})\right)I \\ &= 2^{\mp q_0}\exp\left(\frac{1}{2}\left[q_{\rm L}^{(+)}(h_{-1/2}),\,q_{\rm L}^{(-)}(h_{-1/2})\right]\right)e^{\pm q_{\rm L}^{(-)}(h_{-1/2})}e^{\pm q_{\rm L}^{(+)}(h_{-1/2})}\,,\end{aligned} \tag{4.63}$$

we find that the expression of eq. (4.56) has singularity for $a=-1/2$, and it implies that the classical solution for $a=-1/2$ can be expressed as a singular gauge transform of the trivial vacuum.

5. Summary and discussion

In this paper, we proved the splitting properties of the delta function. Using the splitting properties of the delta function, we found the marginal solutions with well-defined Fock space expressions related to a U(1) current in CSFT. Then, we constructed the scalar solution with a well-defined universal Fock space expression in CSFT, the theory expanded around which can not be transformed into the original one by the string field redefinition. In addition, the scalar solution can be expressed as a singular gauge transform of the trivial vacuum. Though the non-trivial scalar solution was constructed based on the particular function $h_{-1/2}(w)$, we can find other solutions by looking for the function which make the string field redefinition ill-defined. We have not yet understood whether other solutions connect to each other by gauge transformations, or each solution corresponds to a different vacuum.

We expect that the scalar solution represents the condensation of the tachyon, if various scalar solutions represent a single vacuum. The reasons are the following: First, it is impossible to connect the shifted theory to the original one by the string field redefinition, and so, the classical solution represents the non-trivial background which is not merely pure gauge. Secondly, the classical solution is scalar and has a universal expression. Thirdly, the physical states of the original theory are no longer physical in the expanded theory around the classical solution. Indeed, we can find that $Q'_{\mathrm{B}}\,|\mathrm{phys}\rangle \neq 0$ for all states $|\mathrm{phys}\rangle$ such that $Q_{\mathrm{B}}\,|\mathrm{phys}\rangle = 0$.

Of course, in order to clarify this conjecture, we must prove at least two propositions. First, there exists no BRS singlet state for the shifted BRS charge in the Hilbert space [23], or equivalently, the new BRS charge has vanishing cohomology. Secondly, the potential height $S[\Psi_0]$ is equal to the D-brane tension[24]. At present, we can not deny the possibilities to prove these propositions. It should be noted about the latter proposition. As in the case of the marginal solutions, we find that

$$\frac{d}{da}S[\Psi_0] = \int (Q_{\mathrm{B}}\Psi_0 + \Psi_0 * \Psi_0) * \frac{d\Psi_0}{da} = 0\,. \tag{5.1}$$

So, the potential height $S[\Psi_0]$ is equal to zero for $a > -1/2$. However, it may become non-zero value at $a = -1/2$, because the classical solution is ill-defined for $a < -1/2$ and so Ψ_0 is not differentiable at $a = -1/2$.

We should comment on the relation between our scalar solution and a purely cubic theoretical approach to the tachyon vacuum [25]. Their argument was based on purely cubic string field theory (PCSFT) [26]. However, the operator of Q_{L}^2 appeared in the equation of motion in PCSFT has a midpoint singularity, which originates in $\delta(\pm i, \pm i)$ of the surface term in eq. (4.7). Our scalar solution dose not suffer from the midpoint singularity because such a singular surface term disappears due to the condition $f(\pm i) = g(\pm i) = 0$. If we find any classical solution in PCSFT, we should formulate it by other language without midpoint singularity instead of oscillator expansions, which is required also for light-cone type string field theories [19].

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A. Laurent expansion of $h_a(w)$

First, we find the following equation,

$$\frac{\sin x}{\cosh A - \cos x} = 2\sum_{n=1}^{\infty} e^{-nA}\sin(nx) \qquad (A>0)\,. \tag{A.1}$$

Integrating this equation, we obtain

$$\log(\cosh A - \cos x) = A - \log 2 - 2\sum_{n=1}^{\infty}\frac{e^{-nA}}{n}\cos(nx) \qquad (A>0)\,, \tag{A.2}$$

where the integral constant is given by comparing the values at $x=0$ or $\pi/2$. This equation is valid also for the case of $A=0$.

If $a>0$, we can rewrite $h_a(w)$ as

$$\begin{aligned} h_a(w) &= \log\left(1+\frac{a}{2}\left(w+\frac{1}{w}\right)^2\right) \\ &= \log(1+2a\cos^2\sigma) \\ &= \log a + \log\left[\left(1+\frac{1}{a}\right) - \cos(\pi-2\sigma)\right], \end{aligned} \tag{A.3}$$

where we take $w=\exp(i\sigma)$. Applying eq. (A.2) to the second term of eq. (A.3) by taking $A=\log[(1+a+\sqrt{1+2a})/a]$, we find that

$$\begin{aligned} h_a(w) &= \log\frac{1+a+\sqrt{1+2a}}{2} - \\ &\quad -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{1+a-\sqrt{1+2a}}{a}\right)^n\cos(2n\sigma). \end{aligned} \tag{A.4}$$

If $0>a\geq -1/2$, $h_a(w)$ can be rewritten by

$$h_a(w) = \log(-a) + \log\left[\left(-1-\frac{1}{a}\right) - \cos(2\sigma)\right]. \tag{A.5}$$

Applying eq. (A.2) by taking $A=\log[(1+a+\sqrt{1+2a})/(-a)]$, we have the same expansion form of eq. (A.4) as the case of $a>0$.

Taking the limit $a\to 0$ in the right hand-side of eq. (A.4), the result becomes zero and we find that eq. (A.4) holds in the case of $a=0$. Thus, we derive the formula of eq. (4.40).

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